Name: ______________________

UW Math Circle

Week 20 - Three Player Games

1 Turn Based Games

Try some classic two player games, and see what happens when we add in a third player. In each game, determine if there is a strategy any player can follow that will ensure they win every time, no matter what strategy the other players choose.

1. **Pile of Gold:** There are two players and five coins. On each turn, a player may choose to take one or two coins from the middle. The player to takes the last coin wins.

   Play a few rounds of this game. If playing optimally, who will win? Is there a strategy they can use to be impossible to beat?

2. Try the same game but using 10 coins. Who will win, and what is their strategy?

3. **Now add a third player.** With 5 coins and three players, follow the same rules outlined in problem 1. Play a few rounds. Who will win? Is there an optimal strategy? Why or why not?

4. In the last game, there was some ambiguity of how many coins to take if it is impossible to win. Let's add a secondary goal. If it is impossible to be the last to take a coin, you must try to be the second to last to take a coin. The player who does this will win "second place."

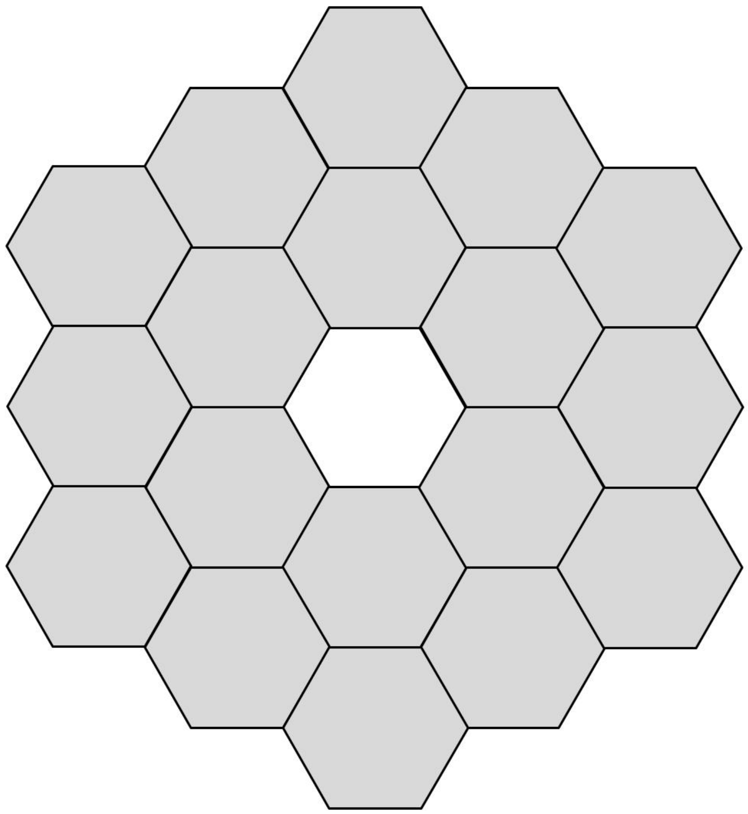
   Following this strategy, determine which player will get first and second place with the number of coins listed in the table:

# of coins	First Place	Second Place
1	Player 1	(none)
2		
3		
4		
5		
6		
7		
8		
9		
10		

   What will the result be for three players and n coins? Why?

Keep the Ring: On a double-layer ring of hexagons, players take turns removing one hexagon until the ring becomes disconnected. The player who breaks the ring loses.

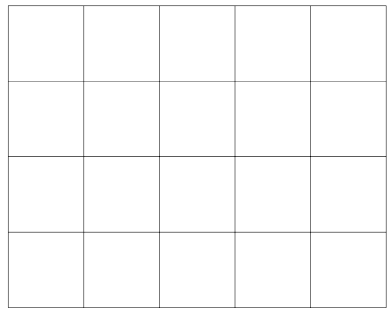
5. In a two player game of Keep the Ring, if both players play optimally who will win? What is their stragtegy?



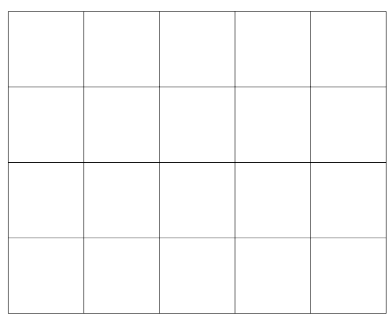
6. Now add a third player. Can two players employ a strategy to ensure that neither of them lose? Which players and what is their strategy? Bonus: is it possible for a different subset of players to team up and always win?

Cram: Players take turns placing a 2x1 domino on a 4x5 grid. The last player to place a valid domino wins.

7. In a two player game, who wins? Is there an optimal strategy?

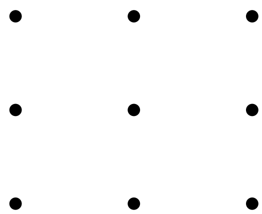


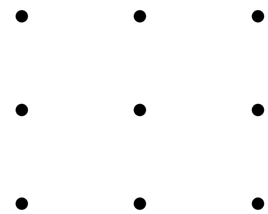
8. In a three player game, who wins? Is there an optimal strategy?



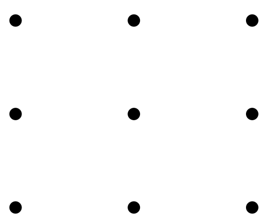
Dots and Boxes: On a grid of dots, players take turns connecting two adjacent dots (horizontally or vertically). Any player who completes a box gets a point (keep track by writing your initial in the completed box) and another turn. Keep playing until all boxes are formed. The player with the most boxes wins!

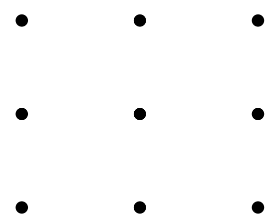
9. Try playing dots with two players on a three by three grid. Is there a winning strategy?



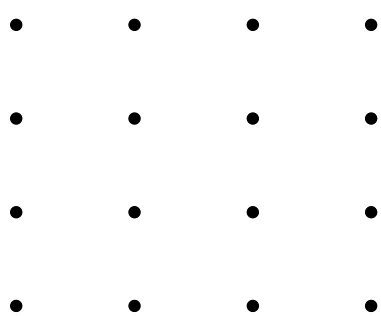


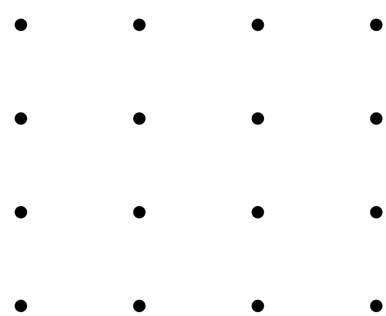
10. Add a third player. Is there a winning strategy?





11. What about two players on a 4x4 grid? Three players?





Sprouts: Start with some dots on the page. Players take turns connecting two dots or drawing a loop connecting a dot to itself, and then draws a new dot in the middle of the line they drew. Lines have to follow these two rules:

- A line cannot intersect another line or itself.
- Each dot can have at most three ends emerging from it. (A loop on a single point thus contributes two ends.)

The game ends when no more lines are possible, with the player who draws the last line declared the winner.

12. Starting with two players and a single dot, who will win?

13. Starting with two players and two dots, who will win? Is there an optimal strategy?

14. Starting with three players and two dots, who will win?

2 Positive Sum Games

Let's move on to a style of games without a single winner. Each player is trying to get the maximum points possible, regardless of how many points the other players receive.

15. Two participants on a game show have the opportunity to win up to \$100. They sit at a table with a divider between them and cannot communicate. Each tells the host whether they want to 'share' or 'steal'.

    If they both share, each gets \$50. If one steals and the other shares, the one who steals gets \$100 and the other gets \$0. If they both choose to steal, they each get \$5.

    If both participants want to maximize their winnings, what should be their strategy?

16. There are three farmers that share a grazing field, and each wants to have three sheep. The field is only large enough to sustain seven sheep. The more sheep share the field, the less healthy all sheep will be.

    - Each healthy sheep is worth 5 points.
    - If there are 8 sheep total, each sheep will be worth 4 points.
    - If there are 9 sheep total, each sheep will be worth 3 points.

    The farmers choose how many sheep (up to three) to have independently and act in their own interests. How many sheep should each farmer have?

Payoff Tables. We can organize the information of a game into a table where each box describes a possible state, and contains the payoffs for each player. Below is a table describing the game show from problem 15.

		Player 2	
		Share	Steal
Player 1	Share	$(50, 50)$	$(0, 100)$
	Steal	$(100, 0)$	$(5, 5)$

We can describe a three player game with a table for each decision player 3 can make. For example, these tables describe the game in problem 16.

Farmer 3:		Farmer 2	
	2 Sheep	2 Sheep	3 Sheep
Farmer 1	2 Sheep	$(10, 10, 10)$	$(10, 15, 10)$
	3 Sheep	$(15, 10, 10)$	$(12, 12, 8)$

Farmer 3:		Farmer 2	
	3 Sheep	2 Sheep	3 Sheep
Farmer 1	2 Sheep	$(10, 10, 15)$	$(8, 12, 12)$
	3 Sheep	$(12, 8, 12)$	$(9, 9, 9)$

A state is called an *equilibrium* state if for any player, changing their decision while the others' remain the same would not benefit that player. For example, in the table for problem 15 we have the equillibrium state (Steal, Steal), and for problem 16 we have (3 Sheep, 3 Sheep, 3 Sheep). In general, there may be more than one equillibrium state.

17. Three oysterers know about a secret patch of 24 oysters. Right now, the oysters are worth \$3 each. If they wait a month, the oysters will grow and be worth \$5 each.

    Each oysterer makes one trip, either now or in a month. Those who goes now can split 18 of them evenly. Anyone who goes in a month can split the remaining oysters evenly.

    Fill out a table describing the possible states. What we the equillibrium states?

Player 3:		Player 2	
	Now	Now	Wait
Farmer 1	Now	(, ,)	(, ,)
	Wait	(, ,)	(, ,)

Player 3:		Player 2	
	Wait	Now	Wait
Player 1	Now	(, ,)	(, ,)
	Wait	(, ,)	(, ,)

18. Come up with a three player game where each player has two options that has no equilibrium state. Is there an optimal strategy?

Player 3:		Player 2	
E		C	D
Farmer 1	A	(, ,)	(, ,)
	B	(, ,)	(, ,)

Player 3:		Player 2	
F		C	D
Player 1	A	(, ,)	(, ,)
	B	(, ,)	(, ,)

19. Let's consider the game show descibed in problem 15, but this time we lift the divider. The players, if they both opt to, may decide to form a *coalition*. This means that they agree on a strategy together (which they must stick to). The coalition only holds if both players agree to the strategy, otherwise they are on their own.

    If both participants are still trying to maximize their winnings, what should be their stategy? Will they form a coalition or not?

20. Look back at the oysterers in problem 17. If we allow player 1 and player 2 to form a coalition, would their strategy change? What if we allow all three to form a coalition?

21. Here is an asymmetric game defined by the following table:

Player 3:		Player 2	
	E	C	D
Farmer 1	A	$(2, 2, 3)$	$(1, 2, 0)$
	B	$(1, 1, 4)$	$(1, 2, 0)$

Player 3:		Player 2	
	F	C	D
Player 1	A	$(1, 2, 0)$	$(1, 2, 0)$
	B	$(1, 2, 0)$	$(3, 4, 1)$

Is there an optimal strategy for each player? Consider all possible coalitions. Would any result in a different optimal strategy?

22. Write your own three player game that is has a single optimal strategy where all payoffs are between 0 and 5. We will challenge your classmates to play this game. Feel free to give players more than two options!